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# NUCLEOCHRONOLOGY AND CHEMICAL EVOLUTION

BEATRICE M. TINSLEY*
The University of Texas at Dallas and Austin
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## ABSTRACT

Considerations of chemical evolution have been used to generalize Schramm and Wasserburg's formalism for deriving a mean age of the elements in the Galaxy at the time ($T$) when the solar system formed.

Comparison of the equations of nucleochronology with those of chemical evolution reveals that the earlier result is restricted to models in which the stellar birthrate varies linearly with the mass of gas in the system, and moreover to those in which the ratio of birthrate to gas mass defines a time-constant much greater than $T$. Relaxing the latter assumption, in particular (which seems necessary in the light of independent evidence), considerably increases the model-dependence of any determination of $T$ from nucleochronometers. It is found that Schramm and Wasserburg's quantity $\Delta^{\max} - \Delta$ (which can in principle be evaluated from abundances and nuclear data on the radioactive elements) does not necessarily lie between $T/2$ and $T$, as it does in the restricted set of models; instead, it may lie anywhere between 0 and $T$, depending critically on the nature of the evolutionary model. In particular, if inflows of metal-poor gas have significantly affected chemical evolution in the solar neighborhood, $T$ could be considerably greater than $\Delta^{\max} - \Delta$. This quantity is shown to be equal to the mean age of stable elements in the gas at time $T$.

It is concluded that the nucleochronometers provide a model-independent *lower* limit to the time $T$, but that derivation of a more precise age of the Galaxy from radioactive time scales will require detailed understanding of its chemical evolution.

*Subject headings:* abundances — Galaxy, the — nucleosynthesis

## I. INTRODUCTION

Cosmochronological quantities such as the ages of radioactive elements, ages of globular clusters and old disk stars, and the expansion age of the Universe are intricately related through numerous parameters describing stellar and chemical evolution in galaxies. Thus a truly model-independent value for one of the times involved has become something of a Holy Grail in this field.

Several attempts have been made to estimate the age of the Galaxy at the time the solar system formed ($T$), from the abundances of long-lived radioactive elements and as few theoretical assumptions as possible. The question of model dependence has been discussed particularly by Fowler (1972) and Schramm and Wasserburg (1970; to be referred to as SW), who found that abundance-to-production ratios and lifetimes of long-lived pairs of elements provide a model-independent "mean age" of the elements at time $T$. The relationship of the mean age to the time $T$ itself is the subject of this paper.

Specifically, SW define an age parameter

$$\Delta_{ij}{}^{\max} \equiv \frac{\ln R(i,j)}{\lambda_i - \lambda_j}\,, \tag{1}$$

where $\lambda_i$, $\lambda_j$ are the decay constants of two elements;

$$R(i,j) \equiv \frac{P_i/P_j}{N_i(T+\Delta)/N_j(T+\Delta)}\,; \tag{2}$$

$P_i/P_j$ is the relative yield of the elements during nucleosynthesis in a generation of stars; $N_i$, $N_j$ are the numbers of atoms of each element present in the proto-solar system material; and $\Delta$ is the postulated interval of "free decay" between the last nucleosynthesis event that enriched the material and its solidification into the meteorites. At time $t$, the rate at which atoms of elements $i$ are being synthesized and ejected into the interstellar gas is $P_i\psi(t)$, where $\psi(t)$ is the mass of stars formed per unit time; if elements $i$ and $j$ are always made in the same stars (or in stars that are always formed in the same proportions), then $P_i/P_j$ is constant. (This assumption will be adopted in the following analysis, so our slight change of notation from SW's will make no difference.) SW's significant result is that in the long-lived limit, where $\lambda_i T \ll 1$ and $\lambda_j T \ll 1$, $\Delta_{ij}{}^{\max}$ must tend to the same value for all pairs, given by

$$\Delta^{\max} - \Delta = T - \langle t \rangle\,, \tag{3}$$

where the mean time $\langle t \rangle$ is, in our notation,

$$\langle t \rangle = \int_0^T t\psi(t)\,dt \Big/ \int_0^T \psi(t)\,dt\,. \tag{4}$$

It is thus possible to evaluate a model-independent mean age of the elements, measured backward from time $T$, since $\Delta_{ij}{}^{\max}$ can be evaluated from the abundance-to-production ratios and decay constants of a long-lived pair while $\Delta$ can be derived from short-lived isotopes (SW; Schramm 1974). Of course, as SW point out, to derive the value of $T$ itself, we need to know $\langle t \rangle/T$, which *is* model-dependent. But if $\psi(t)$

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was decreasing during the interval $(0, T)$, $\langle t \rangle$ lies between 0 and $T/2$, so $\Delta^{\max} - \Delta$ gives $T$ within a factor 2.

The present study was motivated by two questions. First, how important is the caveat at the end of § I of SW's paper, that their rather natural simplifying assumptions might exclude some plausible models from the general validity of their result? And second, is it possible to reduce SW's factor of 2 uncertainty by generalizing Dicke's (1969) exploitation of data on stellar metallicities as a source of information on the rate of nucleosynthesis?

In § II, mathematical results are derived which enable such questions to be studied in some generality. Two specific models, each consistent with other key constraints on chemical evolution, are studied in § III. The general results and examples show, unfortunately, that the answer to the first question is so discouraging that in spite of constraints introduced by stellar metallicities, the uncertainty referred to is a great deal *more* than a factor of 2. A general relation of the *form* of SW's (eq. [3] above) is found to hold; but even with steadily declining stellar birthrates, $T$ can have any value greater than $\Delta^{\max} - \Delta$. The only model-independent quantity that can be obtained is thus a lower limit to the value of $T$. The main results are summarized in § IV, and the effects of various simplifying assumptions are discussed.

## II. GENERAL THEORY

### *a*) *Equations for Chemical Evolution*

Since instantaneous recycling is an excellent approximation for models known to be consistent with evolution of the solar neighborhood, it will be used here (Tinsley 1974 and references therein). Following the notation of that paper, we let $m_{\text{tot}}$ = total mass of the region considered; $m_g$ = mass of "gas" (interstellar matter) in the region; $\psi$ = stellar birthrate, mass per unit time; $f$ = net rate of inflow of gas into the region (usually called "infall"); $m_Z$ = mass of "metals" (stable elements with atomic mass $A \geq 12$) in the gas; $Z$ = metal abundance of the gas = $m_Z/m_g$; $R$ = mass fraction of a stellar generation returned to the gas; $y_Z$ = yield of metals = mass of new metals ejected per unit net increase in the mass of stars in the system.

For a radioactive element with decay constant $\lambda$, let $m_X$, $X$, $y_X$ be defined similarly to the quantities for metals, and let $N$ = number of atoms of the element in the gas = $m_X/AM_{\text{H}}$, where $A$ is its atomic mass and $M_{\text{H}}$ is the proton mass.

The net inflow is expected to be at least metal-poor compared with the ambient gas, so its qualitative effects will be investigated by assuming $Z = X = 0$ in the infalling gas. A further assumption is that stars formed at time $t$ have the metal abundance of the gas, $Z(t)$. Possible effects of these simplifications will be referred to in § IV.

With all of these definitions and assumptions, the basic equations for chemical evolution follow as in Tinsley (1974):

$$dm_{\text{tot}}/dt = f\,, \tag{5}$$

$$dm_g/dt = -\psi(1 - R) + f\,, \tag{6}$$

$$dm_Z/dt = -Z\psi(1 - R) + y_Z\psi(1 - R)\,, \tag{7}$$

$$dm_X/dt = -\lambda m_X - X\psi(1 - R) + y_X\psi(1 - R)\,. \tag{8}$$

Alternative equations for the abundances are clearly

$$m_g dZ/dt = y_Z\psi(1 - R) - Zf\,, \tag{9}$$

$$m_g dX/dt = -\lambda X m_g + y_X\psi(1 - R) - Xf\,, \tag{10}$$

$$\begin{aligned} dN/dt &= -\lambda N + [y_X(1 - R)/AM_H]\psi - [\psi(1 - R)/m_g]N \\ &= -\lambda N + P\psi + \omega_g N\,. \end{aligned} \tag{11}$$

The second line of equation (11) has been written in the form adopted by SW, so that their parameters $P$ and $\omega_g$ can be identified. Evidently,

$$P = y_X(1 - R)/AM_{\text{H}}\,, \tag{12}$$

and

$$\omega_g = -\psi(1 - R)/m_g\,. \tag{13}$$

The yields and $R$ depend on the initial mass function for star formation, so they may be time-dependent. We restrict ourselves here to only one type of variation, imitating a possible initial burst of massive stars leading to prompt enrichment of the gas to abundances $Z_0$, $X_0$, $N_0$. This effect is consistent with, although not demanded by, the compositions of the oldest disk stars (Tinsley 1974, 1975, and references therein). It is most reasonable to assume that the *relative* yields were the same in the initial burst as they are now, so in equations (7)–(11) we make the changes (to avoid additional notation),

$$y_Z \to y_Z[1 + C\delta(t)]\,,\ y_X \to y_X[1 + C\delta(t)]\,,$$
$$P \to P[1 + C\delta(t)]\,, \tag{14}$$

where $y_Z$, $y_X$, $P$, and $C$ are now constants.

### *b*) *The Quantity* $\omega_g$

Equation (13) shows that $\omega_g$ will generally be negative, so it is useful to define a positive quantity,

$$\omega \equiv -\omega_g = \psi(1 - R)/m_g\,, \tag{15}$$

where, by equation (6),

$$\omega = -\frac{1}{m_g}\frac{dm_g}{dt} + \frac{f}{m_g}\,. \tag{16}$$

(It would be possible for $\omega$ to become negative in conditions far from instantaneous recycling such that the rate at which stars were shedding gas, $R(t)\psi(t)$, currently exceeded the rate of star formation, $\psi(t)$.)

SW's solutions to equation (11) are based on the assumption that $\omega$ is constant, which by equation (15)

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would be the case only if the birthrate varied linearly with the gas mass. It will be important to see how the results are affected if $\omega$ is allowed to vary, since it is likely to do so—even any dependence of $\psi$ on the gas *density* might be extremely weak (e.g., Quirk and Tinsley 1973).

The present value of $\omega$ can be roughly estimated from equation (15). Taking $\psi \sim 0.02 m_{\mathrm{tot}}/10^9$ yr, $m_g \sim 0.1 m_{\mathrm{tot}}$, $R \sim 0.3$, we find $\omega \sim 0.14\,(10^9\ \mathrm{yr})^{-1}$, and that now $\omega t \sim 1.5$. This is unlikely to be an order of magnitude different at time $T$. So in spite of the considerable uncertainties in this estimate, it is clearly unsafe to assume that $\omega T \ll 1$. This assumption is implied in SW's derivation of equation (3) (cf. their eqs. [8] and [9]), so a further important question will be as to how the generality of (3) is affected if $\omega T$ is not negligible.

Luckily, SW's derivation of a model-independent formula for $\Delta$ is not affected, since in the short-lived limit, $\lambda T \gg 1$, and when $\lambda \gg \omega$ the effects of $\omega$ are negligible.

### *c*) *The General Solution for Long-lived Elements*

A solution to equation (11) can be written in the general case where $\omega$ may be neither constant nor small. We define the quantity

$$\nu(t) \equiv \int_0^t \omega(t')dt' , \tag{17}$$

which by equation (16) is

$$\nu(t) = \ln \frac{m_{g0}}{m_g} + \int_0^t \frac{f}{m_g}\, dt , \tag{18}$$

where $m_{g0} \equiv m_g(0)$. Then equation (11), with (14), has the solution

$$N(t)e^{\lambda t + \nu(t)} = N_0 + P \int_0^t \psi(t')e^{\lambda t' + \nu(t')}dt' ,$$

where $N_0 = CP\psi(0) \equiv CP\psi_0$. Thus the abundance at time $T + \Delta$, after a period $\Delta$ of free decay, is

$$N(T + \Delta) = Pe^{-\lambda t - \nu(T) - \lambda\Delta}[C\psi_0 + \int_0^T \psi(t)e^{\lambda t + \nu(t)}dt] .$$

Now it is useful to define a mean time

$$t_\nu \equiv \int_0^T t\psi(t)e^{\nu(t)}dt/D , \quad \text{where } D \equiv \int_0^T \psi(t)e^{\nu(t)}dt , \tag{19}$$

and by the mean value theorem $0 < t_\nu < T$. In the long-lived limit, $\lambda T \ll 1$ and $e^{\lambda t} \approx 1 + \lambda t\ (0 \leq t \leq T)$, so the solution reduces to

$$N(T + \Delta) \approx Pe^{-\nu(T) - \lambda\Delta}(C\psi_0 + D)\left[1 - \lambda T + \frac{\lambda t_\nu}{1 + C\psi_0/D}\right] .$$

The final step is to consider two long-lived elements, $i$ and $j$, and to calculate $\Delta_{ij}^{\max}$ from equations (1) and (2). The result is independent of the choice of elements ($\lambda_i$, $\lambda_j$, $P_i$, $P_j$), and is

$$\Delta^{\max} - \Delta = T - \bar{t} , \tag{20}$$

where

$$\bar{t} = t_\nu/(1 + C\psi_0/D) . \tag{21}$$

Thus we still have a result in the useful *form* of SW's (eq. [3]), but the "mean age," $T - \bar{t}$, is redefined. As before, the known quantity $\Delta^{\max} - \Delta$ lies in the interval $(0, T)$ while its exact value is model-dependent. But unfortunately, because the integrand in equation (19) for $D$ is not generally monotonic in time, we cannot any longer be sure of the mean within a factor two, but can only say that $0 < \bar{t} < T$; so the only firm constraint is

$$T > \Delta^{\max} - \Delta .$$

This discrepancy with SW arises not so much because $\omega$ has been allowed to vary with time, as because their tacit assumption $\omega T \ll 1$ has been dropped. To see this, consider the solution in the case corresponding to SW's model with $\omega =$ constant and $C = 0$. Then from equations (17) and (19)–(21), $\nu(t) = \omega t$, $D = \int_0^T \psi(t)e^{\omega t}\,dt$, and

$$\bar{t} = t_\nu = \int_0^T t\psi(t)e^{\omega t}dt/D ,$$

which is clearly not equal to the mean time $\langle t \rangle$ defined by equation (4) and used in (3) unless $|\omega T| \ll 1$.

### *d*) *The Mean Age of Stable Elements*

The mean age $T - \bar{t}$ which occurs in equation (20) is just the mean age of stable elements present in the gas at time $T$, as intuitively expected. This is proved in the Appendix.

It would be very useful to be able to use this fact to evaluate $T - \bar{t}$ directly from the (stellar age, mean abundance) curve, $Z(t)$, along the lines of Dicke (1969). Unfortunately, this cannot be done because the *rate of nucleosynthesis* ($y_Z\psi$) is related only in a model-dependent way to the *rate of enrichment* ($dZ/dt$). Metal-poor infall, in particular, can greatly affect the relation between these rates, and hence the age distribution inferred from a given function $Z(t)$ (Fowler 1972; Searle 1972; Larson 1972). The models discussed in § III will illustrate this problem. Of course, if stellar ages were really accurately determined, their upper bound would give the age of the Galaxy directly!

## III. ILLUSTRATIVE EXAMPLES

### *a*) *Models with Constant $\omega_g$ and No Inflow*

The mathematically simplest models are those with $\omega =$ constant and $f = 0$. Equations (9) and (14)–(16)

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show that $m_g = m_{tot}e^{-\omega t}$, $\psi = \psi_0 e^{-\omega t}$, and $Z = Z_0 + \omega y_Z t$, where $\psi_0 = m_{tot}\omega/(1-R)$ and $Z_0 = C\omega y_Z$. If there is no infall, the stellar metallicity distribution cannot be explained unless $Z_0 \gtrsim 0.15\, Z_\odot$ (Tinsley 1975). Setting $Z_\odot = Z(T)$ (even though the Sun may not have the mean composition of the gas at time $T$), we have the relation $C = Z_\odot/\omega y_Z - T$. Note that this model has an exponentially declining rate of nucleosynthesis, but a constant rate of enrichment.

The mean ages, derived from equations (4), (19), and (21), are

$$\langle t\rangle = \frac{1 - e^{-\omega T}(1 + \omega T)}{\omega(1 - e^{-\omega T})}\,, \qquad \bar{t} = \frac{T/2}{1 + Z_0/\omega y_Z T}\,.$$

Only in the limit $\omega T \ll 1$ does $\langle t\rangle = \bar{t}$.

Plausible estimates of the parameters are: $\omega = 0.15(10^9\ \mathrm{yr})^{-1}$ (§ II*b*), $Z_\odot/y_Z = 1$ (Talbot and Arnett 1973*a*), $\Delta^{max} - \Delta = 3 \times 10^9$ yr (Schramm 1974). With these values, we have according to this model $\bar{t} = 1.6 \times 10^9$ yr, $\langle t\rangle = 2.0 \times 10^9$ yr, $Z_0 = 0.32\, Z_\odot$, and $T = 4.6 \times 10^9$ yr.

### *b) Models with Metal-free Inflow and Constant Gas Mass*

This case is a convenient approximation to plausible models for chemical evolution based on a variety of physical assumptions (cf. references cited in Tinsley 1974, 1975). Effects of infall on nucleochronology have been discussed by Fowler (1972).

A useful parameter in the analysis is $\nu(t)$ defined by equation (17). Since $m_g$ is constant, (5) and (16) give $\nu(t) = m_{tot}(t)/m_g - 1$, while $\omega = f/m_g = d\nu/dt$. Also, by equation (6), $f(t) = \psi(t)(1 - R)$. For the metal abundance, equations (9) and (14) now give

$$Z(t) = Z_0 e^{-\nu(t)} + y_Z(1 - e^{-\nu(t)})\,,$$

where $Z_0 = y_Z C\psi_0(1 - R)/m_g$. At the present time, $\nu \sim 10$, so the model predicts $Z \sim y_Z$, in excellent agreement with the theoretical estimate $y_Z \sim 0.02$ (Talbot and Arnett 1973*a*).

Again the mean ages can be derived from equations (4), (19), and (21), with the results,

$$\langle t\rangle = \frac{1}{\nu(T)}\int_0^T t\,d\nu\,,$$

$$D = \frac{m_g}{1 - R}(e^{\nu(T)} - 1)\,, \qquad t_\nu = \int_0^T te^\nu d\nu/(e^{\nu(T)} - 1)\,,$$

$$\bar{t} = \frac{\int_0^T te^\nu d\nu}{e^{\nu(T)} - 1 + Z_0/y_Z}\,. \qquad (22)$$

The last expression shows that if the total mass increase due to infall occurred over a long time-scale, $\bar{t}/T$ would approach unity; thus, from equation (20), $T$ would be much greater than $\Delta^{max} - \Delta$, the mean age of elements in the gas. As discussed by Fowler (1972), this model can lead to a very slow rate of enrichment, in spite of continuous nucleosynthesis.

As a numerical example, consider a model with $f$ = constant and $Z_0 = 0$ (which *is* compatible with the stellar metallicity distribution if there is infall [Tinsley 1974, 1975 and references therein]). Then $\omega = f/m_g$ = constant, and $\nu = \omega t$. Estimating $\nu \approx 9$ at the present time (since $m_g \approx 0.1 m_{tot}$ now), we have $\omega(T + 4.6 \times 10^9$ yr) = 9. Adopting as in § III*a* the estimate $\Delta^{max} - \Delta = 3 \times 10^9$ yr, we find from the above equations that (20) is satisfied with the following values: $\omega = 0.33(10^9\ \mathrm{yr})^{-1}$ (agreeing with the estimates of $\omega$ and $f/m_g$ within their uncertainties), $\bar{t} = 19.4 \times 10^9$ yr, $\langle t\rangle = 11.2 \times 10^9$ yr, and $T = 22.4 \times 10^9$ yr! Since $f$ is more likely to be a decreasing function of time than constant, such a great age might be regarded as an upper limit for realistic models. (Incidentally, the estimate of $\Delta^{max}$ from $^{232}Th/^{238}U$ cannot be used if the age is really so great, since neither element is then "long-lived," but the $^{187}Re/^{187}Os$ chronology could be used [Schramm 1974].)

## IV. CONCLUSIONS

The possibility of estimating the age $T$ of the Galaxy at the time of formation of the solar system has been investigated along the lines introduced by Schramm and Wasserburg (1970), but with greater generality. A formula (20) has been derived, which like their result (3) relates $T$ to the quantity $\Delta^{max} - \Delta$ that is obtainable from abundance-to-production ratios and decay constants of radioactive elements (Schramm 1974). The present formula is valid in two circumstances not previously covered: one is that the quantity $\omega$ (eq. [15]), which is just the net stellar birthrate per unit mass of gas, need not be constant; the other is that $\omega T$ need not be much smaller than 1. The latter condition was tacitly assumed by SW, but it is unlikely to be valid (§ II*b*)—note that it would imply a time scale for turning gas into stars much less than $T$—and it has an important effect on the usefulness of equation (20). Specifically, it has been shown here (§ II*c*) that $\Delta^{max} - \Delta$ provides in general only a *lower limit* to the age $T$. Any closer relationships are very model-dependent, so they cannot be trusted until independent evidence lends conclusive support to a particular model for the evolution of the Galaxy in the solar neighborhood. It is especially important to know the extent, if any, of prompt initial enrichment of the interstellar gas in heavy elements, and the rate, if any, of inflow of metal-poor gas at all epochs prior to $T$.

Using Schramm's (1974) estimates, we can conclude that the lower limit to $T$ lies between 1.4 and 5.3 billion years, so the present age of the Galaxy has a lower limit between 6 and 10 billion years.

It has been shown that $\Delta^{max} - \Delta$ is in fact just the mean age of stable elements present in the gas at time $T$.

Several simplifying assumptions have been used in the present analysis, and their effects should be assessed. (*a*) The possibility of metal-enhanced star formation (Searle 1972) has been neglected. Talbot and Arnett (1973*b*) show that this process leads to a lower mean age of the elements in the gas in a given

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model, so it leads to longer estimates of $T$. Thus the above lower limits are not affected. (*b*) The only time variations of the yields of heavy elements that have been considered are an initial increase of all yields in the same proportions, to give prompt initial enrichment. Reeves and Johns (1974) have been able to use the nucleochronometers themselves to set some limits on possible variations of *relative* yields; but in view of the many uncertainties involved and the model-dependence of their criteria, no strong constraints can be set. (*c*) The instantaneous recycling approximation introduces systematic errors in the chronologies by neglecting the time elements are stored in stellar envelopes before release. An exact calculation would therefore show a greater mean age for elements released from dying low-mass stars, thus decreasing $\bar{t}$ and in turn the estimate of $T$. The above lower limits should be reduced to allow for this effect, but the difference should be slight since one estimates that most of the mass returned from a given generation of stars has been stored for less than $10^9$ yr (Talbot and Arnett 1973*a*, Fig. 2). (*d*) Finite abundances in the infalling gas have not been considered, so the effects of infall found here are more extreme than some models would predict. (*e*) Finally, chemical inhomogeneities in the gas have been ignored. Among the possible errors introduced is the usual problem that meteoritic abundances (used to evaluate $\Delta^{\max} - \Delta$) may not be typical of those in the interstellar medium at time $T$.

A more detailed study that relaxed some of these restrictions would only strengthen the principal conclusion of this paper, which is that nucleochronologies give, at best, only a very model-dependent age estimate for the Galaxy. A "mean" age and a lower age limit, with uncertainties arising from problems other than the evolutionary model, can be derived, but not an upper limit.

It is a pleasure to thank Dr. David N. Schramm for several valuable discussions on the meaning of a "mean age" of the elements, and for his encouragement to pursue the calculations reported here.

## APPENDIX

It will be shown that the quantity $\Delta^{\max} - \Delta$, given by equations (20) and (21), is equal to the mean age of *stable* elements in the gas at time $T$.

Consider the age distribution of metals in the gas at time $t$, and let $p(\tau, t)d\tau$ be the fraction of metals in the gas at time $t$ that were formed in the interval $(\tau, \tau + d\tau)$ measured from $t = 0$. The equation corresponding to (6) for this fraction is

$$d[m_Z p(\tau, t)]/dt = -Z\psi(1 - R)p(\tau, t) + y_Z\psi(1 - R)\delta(t - \tau);$$

or, using equations (6) and (7),

$$\frac{dp(\tau, t)}{dt} = \frac{y_Z\psi(1 - R)}{Zm_g}[\delta(t - \tau) - p(\tau, t)].$$

The $\delta$-function can be eliminated by integrating with respect to $t$, which then gives an expression for $p(\tau, \tau)$ to be used as an initial condition in the simpler equation obtained by differentiating again:

$$\frac{dp(\tau, t)}{dt} = -\frac{y_Z\psi(1 - R)}{Zm_g}p(\tau, t), \qquad \text{with } p(\tau, \tau) = \left[\frac{y_Z\psi(1 - R)}{Zm_g}\right]_{t=\tau}.$$

The solution is

$$p(\tau, t) = \frac{y_Z(\tau)}{Z(\tau)}\omega(\tau)\exp\left[-\int_\tau^t \frac{y_Z(t')\omega(t')}{Z(t')}dt'\right]. \tag{A1}$$

Next we simplify this expression using equation (9), which, with (14), can be written

$$dZ/dt + Zf/m_g = \omega y_Z[1 + C\delta(t)].$$

Defining two new functions

$$\theta(t) \equiv \int_0^t \frac{f(t')}{m_g(t')}dt', \qquad \varphi(t)\int_0^t \omega(t')e^{\theta(t')}dt', \tag{A2}$$

we have the solution

$$Z(t) = Z_0 e^{-\theta(t)} + y_Z e^{-\theta(t)}\varphi(t),$$

where $Z_0 = y_Z C\omega(0) = y_Z C\psi_0(1 - R)/m_{g0}$. Therefore,

$$\frac{y_Z(t)\omega(t)}{Z(t)} = \frac{[1 + C\delta(t)]\omega(t)e^{\theta(t)}}{Z_0 + y_Z\varphi(t)}, \tag{A3}$$

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and the integral of this quantity from 0 to $t$ is simply $1 + \ln [1 + (y_Z/Z_0)\varphi(t)]$. With these results, equation (A1) can be written

$$p(\tau, t) = \frac{\omega(\tau)e^{\theta(\tau)}[1 + C\delta(\tau)]}{Z_0/y_Z + \varphi(t)} .$$

This relation shows that $\int_0^t p(\tau, t)d\tau = 1$, as required by the definition of $p$, and that the mean formation time for metals present at time $T$ is

$$t_Z \equiv \int_0^T \tau p(\tau, T)d\tau = \frac{1}{\varphi(T) + Z_0/y_Z} \int_0^T t\omega(t)e^{\theta(t)}dt . \tag{A4}$$

In order to show that $t_Z$ is the same mean as $\bar{t}$ (eqs. [20] and [21]), we use equations (18), (A2), and (15) to write

$$m_g(t) = m_{g0}e^{\theta(t) - \nu(t)} , \qquad \psi(t) = \frac{m_{g0}}{1 - R}\, \omega(t)e^{\theta(t) - \nu(t)} .$$

Substituting for $\psi(t)$ in equation (19), we find that

$$D = \frac{m_{g0}}{1 - R}\varphi(T) , \qquad t_\nu = \frac{1}{\varphi(T)} \int_0^T t\omega(t)e^{\theta(t)}dt .$$

For the denominator in equation (21) for $\bar{t}$, (A3) shows that $C\psi_0/D = Z_0/y_Z\varphi(T)$. Finally, comparison between the formulae (21) and (A4) now shows that $t_Z$ and $\bar{t}$ are identical.

## REFERENCES


Dicke, R. H. 1969, *Ap. J.*, **155**, 123.
Fowler, W. A. 1972, in *Cosmology, Fusion, and Other Matters, A Memorial to George Gamow*, ed. F. Reines (Boulder: Associated University Press).
Larson, R. B. 1972, *Nature Phys. Sci.*, **236**, 7.
Quirk, W. J., and Tinsley, B. M. 1973, *Ap. J.*, **179**, 69.
Reeves, H., and Johns, O. 1974, preprint.
Schramm, D. N. 1974, *Ann. Rev. Astr. and Ap.*, **12**, 383.
Schramm, D. N., and Wasserburg, G. J. 1970, *Ap. J.*, **162**, 57 (SW).
Searle, L. 1972, in *IAU Colloquium No. 17*, ed. G. Cayrel de Strobel and M. Delplace (Paris: Meudon Observatory).
Talbot, R. J., and Arnett, W. D. 1973*a*, *Ap. J.*, **186**, 51.
———. 1973*b*, *ibid.*, p. 69.
Tinsley, B. M. 1974, *Ap. J.*, **192**, 629.
———. 1975, *ibid.*, **197**, 159.


Beatrice M. Tinsley: Lick Observatory, University of California, Santa Cruz, CA 95064

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